$$\Phi = \Phi(\alpha + \theta) \pm \Phi(2\beta + \alpha - \theta)$$

where $\Phi(\theta)$ is the corresponding solution for the Riemann surface.

These constructions using formulae (3.1), (3.2) may be carried out to obtain the solution of the diffraction problem for a wave $f(t, r, z, \alpha + \theta) H(\eta - \cos(\alpha + \theta))$ by a wedge and a half-plane.

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WEAKLY LINEAR OSCILLATIONS OF THE RADIUS OF A VAPOUR BUBBLE IN AN ACOUSTIC FIELD*

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Non-linear heat-and-mass exchange effects between a vapour bubble and a surrounding liquid under periodic pressure oscillations generated by an acoustic field of length significantly greater than the radius of the bubble are investigated. Based on a closed system of equations for the spherically symmetric processes around an isolated bubble /1/, the method of multiple scales /2, 3/ is used to derive asymptotic equations for the behaviour of the average bubble radius, accurate to the second order in the field amplitude.

Linear and weakly linear oscillations of vapour bubbles in acoustic fields have been studied quite extensively, and the main results have been summarized in the literature /1, 4/. The most comprehensive investigation of the "smoothed heat transfer effect" for vapour bubbles, that is, the variation of the average bubble radius over a large number of periods due to the non-linearity of heat-and-mass exchange, may be found in /4/. This paper departs from previous publications on "smoothed heat transfer" in its systematic allowance for the non-equilibrium conditions of the phase transitions, which, over a certain parameter range, exert a decisive effect on the dynamics of the average bubble radius; the non-uniform vapour temperature in the bubble is also taken into account. In addition, application of the method of multiple scales has justified certain assumptions previously adopted in applications of the averaging method to derive equations for the dynamics of the average bubble radius.

1. Statement of the problem. We shall study the behaviour of a spherical vapour bubble in an unbounded space occupied by an ideal incompressible liquid, with the pressure at infinity p_∞ varying periodically about an equilibrium value $p_* = p_s(T_*)$, $T_* = T_\infty$ (T is the temperature, the subscript s denotes the parameters on the saturation curve and the asterisk denotes the parameters of the unperturbed state):

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$$p_\infty = p_*(1+\varepsilon\varphi(t)),\ \varphi(t)=\varphi(t+2\pi/\omega),\ |\varphi|\leqslant 1 \qquad (1.1)$$

where φ is a periodic function that can be expanded in Fourier series, ω is the angular frequency, t is the time and ε is the relative amplitude of the perturbation, which is assumed to be a small parameter $\varepsilon \ll 1$.

There is a wide range of situations involving non-linear non-equilibrium dynamics of vapour bubbles, in which one can ignore surface tension and the temperature jump across the phase interface; in addition, one can assume that the vapour is an ideal gas, the homobaric condition holds and the processes near the bubble are spherically symmetric /1/. Under these conditions a closed system of equations and boundary conditions for heat-and-mass exchange between bubble and liquid in the variable pressure field may be written as follows /1/:

$$p_g(t) = R_g T_g(r,t)\rho_g(r,t),\ \rho_l = \text{const}$$
$$\rho_l c_l\left(\frac{\partial T_l}{\partial t} + w_l\frac{\partial T_l}{\partial r}\right) = \lambda_l\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_l}{\partial r}\right)$$
$$\rho_g c_g\left(\frac{\partial T_g}{\partial t} + w_g\frac{\partial T_g}{\partial r}\right) = \lambda_g\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_g}{\partial r}\right) + \frac{dp_g}{dt}$$
$$w_l = w_{la}\frac{a^2}{r^2},\quad w_g = \frac{\gamma-1}{\gamma p_g}\lambda_g\frac{\partial T_g}{\partial r} - \frac{r}{3\gamma p_g}\frac{dp_g}{dt} \qquad (1.2)$$
$$w_{la} = \frac{da}{dt} - \frac{\xi}{\rho_l},\quad w_{ga} = \frac{da}{dt} - \frac{\xi}{\rho_{ga}}$$
$$a\frac{dw_{la}}{dt} + \left(2\frac{da}{dt} - \frac{1}{2}w_{la}\right)w_{la} = \frac{p_g - p_\infty}{\rho_l}$$
$$w_{ga} = -\frac{\gamma-1}{\gamma p_g}q_g - \frac{k}{3\gamma p_g}\frac{dp_g}{dt}$$
$$q_g - q_l = \xi l,\quad q_g = -\lambda_g\left.\frac{\partial T_g}{\partial r}\right|_{r=a},\quad q_l = -\lambda_l\left.\frac{\partial T_l}{\partial r}\right|_{r=a}$$
$$\xi = \frac{\beta[p_s(T_a) - p_g]}{\sqrt{2\pi R_g T_a}},\quad p_s(T) = p_*\exp\left[\frac{1}{k_s}\left(1 - \frac{T_*}{T}\right)\right],\quad k_s = \frac{R_g T_*}{l}$$
$$T_g|_{r=a} = T_l|_{r=a} = T_a,\quad \left.\frac{\partial T_g}{\partial r}\right|_{r=0} = 0,\quad T_l|_{r=\infty} = T_*$$

Here r is a radial coordinate, measured from the centre of the bubble; ρ, w, q and ξ are the pressure, radial velocity, heat flow at the boundary and rate of vaporization per unit surface area, a is the bubble radius, $\gamma, R_g, l, c, \lambda$ are quantities assumed here to be constant: the adiabatic exponent of the gas, the gas constant, the specific heat of vaporization, the specific heat (for a gas - at constant pressure) and the thermal conductivity, and $\beta(p_g, T_a)$ is the acommodation coefficient, which is assumed to be a known function of the pressure and temperature. The subscripts g and l refer to the gas and liquid, respectively, the parameters at the phase interface are given the subscript a.

2. *Method of solution.* The space-time transformation $(r,t)\to(\eta,t)$, where $\eta = r/a(t)$ $(\partial/\partial r \to a^{-2}\partial/\partial\eta,\ \partial/\partial t \to \partial/\partial t - (a^{-1}da/dt)\,\eta\partial/\partial\eta)$, reduces the heat Eqs.(1.2) to problems in domains with fixed boundaries. The radial velocities w_l, w_g, w_{ga} and the gas density ρ_g are expressed in terms of the other unknowns, which are expanded in asymptotic series:

$$T_a = T_*(\theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \ldots),\quad T_\alpha = T_*(u_0^{(\alpha)} + \varepsilon u_1^{(\alpha)} + \varepsilon^2 u_2^{(\alpha)} + \ldots) \qquad (2.1)$$
$$a = a_0(1 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots),\quad w_{la} = \varkappa_g a_0^{-1}(w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots)$$
$$p_g = p_*(p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots),\quad q_\alpha = \lambda_\alpha T_* a^{-1}(q_0^{(\alpha)} + \varepsilon q_1^{(\alpha)} + \varepsilon^2 q_2^{(\alpha)} + \ldots)$$
$$\xi = \rho_{g*}\varkappa_g a_0^{-1}(j_0 + \varepsilon j_1 + \varepsilon^2 j_2 + \ldots),\quad \alpha = g, l\quad (\varkappa_\alpha = \lambda_\alpha\rho_{\alpha*}^{-1}c_\alpha^{-1})$$

The procedure for constructing uniformly valid expansions by the method of multiple scales /2/ is similar to that described in /3/. As functions of t, the unknowns are assumed to depend on a sequence of time variables $\{t_k\}$, $t_k = \varepsilon^k t\ (k = 0, 1, 2, \ldots)$ and the operator of differentiation with respect to t is expanded in an asymptotic series:

$$d/dt = \partial/\partial t_0 + \varepsilon\partial/\partial t_1 + \varepsilon^2\partial/\partial t_2 + \cdots \qquad (2.2)$$

The function $\varphi(t)$ of (1.1) is assumed to be a function of the "fast" time t only.

Substituting (1.1), (2.1),(2.2) into (1.2) and collecting terms in like powers of ε, one can show that $u_0^{(\alpha)} \equiv 1,\ q_0^{(\alpha)} \equiv 0\ (\alpha = l, g),\ p_0 \equiv \theta_0 \equiv 1,\ w_0 \equiv j_0 \equiv 0,\ a_0 = a_0(t_1, t_2, \ldots)$, finally obtaining the following linear inhomogeneous systems of equations for the m-th approximation $(m > 0)$:

$$
\begin{gathered}
\frac{a_0^2}{\varkappa_g}\frac{\partial u_m^{(l)}}{\partial t_0}-\frac{k_\varkappa}{\eta^2}\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial u_m^{(l)}}{\partial\eta}\right)=f_m^{(l)} \\
\frac{a_0^2}{\varkappa_g}\left(\frac{\partial u_m^{(g)}}{\partial t_0}-k_\gamma\frac{\partial p_m}{\partial t_0}\right)-\frac{1}{\eta^2}\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial u_m^{(g)}}{\partial\eta}\right)=f_m^{(g)} \\
u_m^{(l)}|_{\eta=1}=u_m^{(g)}|_{\eta=1}=\theta_m,\quad u_m^{(l)}|_{\eta=\infty}=0,\quad \partial u_m^{(g)}/\partial\eta|_{\eta=0}=0 \\
\frac{a_0^2}{\varkappa_g}\frac{\partial a_m}{\partial t_0}-k_\rho j_m-w_m=f_m^{(a)},\quad \frac{a_0^2}{\varkappa_g}\left(\frac{1}{3\gamma}\frac{\partial p_m}{\partial t_0}+\frac{\partial a_m}{\partial t_0}\right)+q_m^{(g)}-j_m=f_m^{(p)} \\
\tau_w\partial w_m/\partial t_0-p_m=f_m^{(w)},\quad j_m-a_0d_\sigma^{-1}(\theta_m-k_s p_m)=f_m^{(\theta)} \\
q_m^{(\alpha)}=-\partial u_m^{(\alpha)}/\partial\eta|_{\eta=1},\quad \alpha=g,l,\quad k_s k_\gamma^{-1}(q_m^{(g)}-k_\lambda q_m^{(l)})-j_m=f_m^{(j)} \\
k_\gamma=1-1/\gamma,\quad k_\rho=\rho_{g*}/\rho_l,\quad k_\lambda=\lambda_l/\lambda_g,\quad k_\varkappa=\varkappa_l/\varkappa_g \\
d_\sigma=\varkappa_g\sqrt{2\pi R_g T_*}/(\beta_* l),\quad \tau_w=\varkappa_g\rho_l/p_*,\quad \beta_*=\beta(p_*,T_*)
\end{gathered}
\tag{2.3}
$$

Here $f_m^{(\alpha)}$ are functions depending on approximations of order less than m ($\alpha=l,g,a,w,p,j,\theta$).

The condition for the existence of a t_0-periodic solution of (2.3), i.e., a solution that can be expanded in Fourier series as a function of the "fast" time, is /3/

$$
x(t_0,t_1,t_2,\ldots)=\operatorname{Re}\left\{\sum_{n=0}^{\infty}x_n^0(t_1,t_2,\ldots)e^{in\omega t_0}\right\}
\tag{2.4}
$$

where x_n^0 is the complex amplitude. This yields a system of the form (2.3) for the complex amplitudes in the m-th approximation, with the factor $i\omega n$ in place of the operator $\partial/\partial t_0$.

On the assumption that the integrals exist, the solution of the inhomogenous heat-transfer problems for the complex temperature amplitudes in the liquid and in the gas is /3/

$$
\begin{gathered}
u_{mn}^{(l)0}=\eta^{-1}[A_{mn}e_n^-(\eta)+G_{mn}^{(l)}(\eta)] \\
u_{mn}^{(g)0}=k_\gamma p_{mn}^0+\eta^{-1}[B_{mn}S_n(\eta)+G_{mn}^{(g)}(\eta)] \\
A_{mn}=\theta_{mn}^0-G_{mn}^{(l)}(1),\quad B_{mn}=(\theta_{mn}^0-G_{mn}^{(g)}(1)-k_\gamma p_{mn}^0)/S_n(1) \\
G_{mn}^{(l)}(\eta)=\frac{1}{2}(k_\varkappa/s_n)^{1/2}\left[e_n^-(\eta)\int_1^\eta e_n^+(x)xf_{mn}^{(l)0}(x)\,dx+\right. \\
\left.e_n^+(\eta)\int_\eta^\infty e_n^-(x)xf_{mn}^{(l)0}(x)\,dx\right] \\
G_{mn}^{(g)}(\eta)=s_n^{-1/2}\int_0^\eta[C_n(\eta)S_n(x)-S_n(\eta)C_n(x)]xf_{mn}^{(g)0}(x)\,dx \\
e_n^\pm(\eta)=e^{\pm(\eta-1)\sqrt{s_n/k_\varkappa}},\quad S_n(\eta)=\operatorname{sh}(\eta\sqrt{s_n}),\quad C_n(\eta)=\operatorname{ch}(\eta\sqrt{s_n}) \\
s_n=i\omega n a_0^2/\varkappa_g,\quad \operatorname{Re}\{\sqrt{s_n}\}\geqslant 0 \\
q_{mn}^{(l)0}=h_n^{(l)}\theta_{mn}^0-k_\varkappa^{-1}F_{mn}^{(l)},\quad h_n^{(l)}=1+(s_n/k_\varkappa)^{1/2} \\
q_{mn}^{(g)0}=-h_n^{(g)}(\theta_{mn}^0-k_\gamma p_{mn}^0)+F_{mn}^{(g)},\quad h_n^{(g)}=\sqrt{s_n}\operatorname{cth}\sqrt{s_n}-1 \\
F_{mn}^{(l)}=\int_1^\infty\eta f_{mn}^{(l)0}e_n^-(\eta)\,d\eta,\quad F_{mn}^{(g)}=[S_n(1)]^{-1}\int_0^1\eta f_{mn}^{(g)0}S_n(\eta)\,d\eta
\end{gathered}
\tag{2.5}
$$

Thus, it follows from (2.3)-(2.5) that in the m-th approximation the unknown complex amplitudes satisfy a linear inhomogeneous algebraic equation:

$$
L_n\mathbf{X}_{mn}=\mathbf{Y}_{mn}
\tag{2.6}
$$

$$
L_n=\begin{Vmatrix}
s_n & -1 & 0 & 0 & -k_\rho \\
0 & \varkappa_g a_0^{-2}s_n\tau_w & -1 & 0 & 0 \\
s_n & 0 & \frac{1}{3}\gamma^{-1}s_n+k_\gamma h_n^{(g)} & -h_n^{(g)} & -1 \\
0 & 0 & a_0d_\sigma^{-1}k_s & -a_0d_\sigma^{-1} & 1 \\
0 & 0 & k_s h_n^{(g)} & -k_s k_\gamma^{-1}(h_n^{(g)}+k_\lambda h_n^{(l)}) & -1
\end{Vmatrix}
$$

$$
\mathbf{X}_{mn}=(\overset{\circ}{a}_{mn},\overset{\circ}{w}_{mn},\overset{\circ}{p}_{mn},\overset{\circ}{\theta}_{mn},\overset{\circ}{j}_{mn})^T
$$

$$
\mathbf{Y}_{mn}=(f_{mn}^{(a)0},f_{mn}^{(w)0},f_{mn}^{(p)0}-F_{mn}^{(g)},f_{mn}^{(\theta)0},f_{mn}^{(j)0}-k_s k_\gamma^{-1}[F_{mn}^{(g)}+k_\lambda k_\varkappa^{-1}F_{mn}^{(l)}])^T
$$

The superscript T denotes transposition.

Previous studies /6, 7/ considered the problem of a steadily pulsating bubble in a monochromatic acoustic field; it was observed that solutions exist at any frequencies $\omega \neq 0$, i.e., if $s_n \neq 0$ we have $\det L_n \neq 0$ and the solution $\mathbf{X}_{mn}$ exists and is unique. If $n = 0$ the matrix of the system becomes singular and $\operatorname{rank} L_0 = 4$. A necessary and sufficient condition for the existence of a solution is then $\operatorname{rank}(L_0 \mid \mathbf{Y}_{m0}) = 4$, where $(L_0 \mid \mathbf{Y}_{m0})$ is the augmented matrix of the system in the m-th approximation with $n = 0$. This condition, when developed, is

$$-(1 + \delta_\sigma a_0^{-1}) f_{m0}^{(p)0} = k_s{}^2 k_\gamma^{-1} k_\lambda f_{m0}^{(w)0} + \delta_\sigma a_0^{-1} f_{m0}^{(\theta)0} - f_{m0}^{(j)0} + k_s k_\lambda (k_\gamma k_\varkappa)^{-1} F_{m0}^{(l)} - (1 - k_s k_\gamma^{-1} + \delta_\sigma a_0^{-1}) F_{m0}^{(g)} \tag{2.7}$$

$$F_{m0}^{(l)} = \lim_{s_n \to 0} F_{mn}^{(l)} = \int_1^\infty \eta f_{m0}^{(l)0}\, d\eta, \quad F_{m0}^{(g)} = \lim_{s_n \to 0} F_{mn}^{(g)} = \int_0^1 \eta^2 f_{m0}^{(g)0}\, d\eta, \quad \delta_\sigma = \frac{k_s k_\lambda}{k_\gamma} d_\sigma$$

Thus the system will have a t_0-periodic solution only if condition (2.7) is satisfied. In that case the solution is unique up to a solution of the homogeneous system (2.3):

$$a_m = a_m(t_1, t_2, \ldots), \quad w_m = p_m \equiv \theta_m \equiv j_m \equiv 0, \quad u_m^{(l)} \equiv 0, \quad u_m^{(g)} \equiv 0$$

3. *First approximation.* It follows from (1.1), (1.2) and (2.1)-(2.3) that to a first approximation ($m = 1$):

$$f_1^{(\alpha)} \equiv 0, \quad \alpha = l, g, \theta, j; \quad f_1^{(a)} = f_1^{(p)} = -a_0 \varkappa_g^{-1} \partial a_0 / \partial t_1; \quad f_1^{(l)} = -\varphi \tag{3.1}$$

The condition for t_0-periodic solutions to exist gives

$$(a_0 + \delta_\sigma)\, \partial a_0 / \partial t_1 = -k_s{}^2 k_\gamma{}^{-1} k_\lambda \varkappa_g \varphi_0{}^\circ \tag{3.2}$$

Integrating with respect to t_1, we get

$${}^1/_2 a_0{}^2 + \delta_\sigma a_0 = -k_s{}^2 k_\gamma{}^{-1} k_\lambda \varkappa_g\, \varphi_0{}^\circ t_1 + C(t_2, t_3, \ldots) \tag{3.3}$$

Assume that condition (3.2) holds. If $n \neq 0$ it follows from (3.1) that the inhomogeneity vector in (2.6) is $\mathbf{Y}_{1n} = (0, -\varphi_n{}^\circ, 0, 0, 0)^T$. Hence, using Cramer's rule, we find that

$$\overset{\circ}{\alpha}_{1n} = -\varphi_n{}^\circ \Delta_n^{(\alpha)} / \Delta_n, \quad \alpha = a, w, p, \theta, j; \quad \Delta_n = \det L_n \tag{3.4}$$

$$\Delta_n^{(a)} = [a_0 d_\sigma^{-1} k_\gamma (1 - k_s k_\gamma^{-1})^2 + {}^1/_3 \gamma^{-1} k_s k_\gamma^{-1} s_n + k_s k_\lambda h_n^{(l)}]\, h_n^{(g)} + k_s k_\gamma^{-1} k_\lambda (a_0 d_\sigma^{-1} k_s + {}^1/_3 \gamma^{-1} s_n)\, h_n^{(l)} + {}^1/_3 \gamma^1 a_0 d_\sigma^{-1} s_n, \quad \Delta_n^{(w)} = s_n \Delta_n^{(a)}$$

$$\Delta_n^{(p)} = -s_n [a_0 d_\sigma^{-1} + k_s k_\gamma^{-1} (h_n^{(g)} + k_\lambda h_n^{(l)})], \quad \Delta_n^{(\theta)} = -s_n k_s (a_0 d_\sigma^{-1} + h_n^{(g)})$$

$$\Delta_n^{(j)} = s_n a_0 d_\sigma^{-1} [k_s k_\gamma^{-1} k_\lambda h_n^{(l)} - (1 - k_s k_\gamma^{-1})\, h_n^{(g)}], \quad \Delta_n = s_n{}^2 a_0^{-2} \varkappa_g \tau_w \Delta_n^{(a)} - \Delta_n^{(p)}$$

The quantity k_ρ in (3.4) is negligible compared with unity in the coefficients $(1 - k_\rho)$.

Eqs.(3.4) are in agreement with those obtained in /6/ for steady oscillations of a vapour bubble in a monochromatic acoustic field. An analysis of the complex amplitude of oscillations of the bubble radius may be found in /1, 4, 7/.

The complex amplitudes of the temperature distributions in the phases may be determined from (2.5) and (3.1):

$$u_{1n}^{(l)\circ} = \overset{\circ}{\theta}_{1n} \eta^{-1} e_n^-(\eta), \quad u_{1n}^{(g)\circ} = k_\gamma \overset{\circ}{p}_{1n} + (\overset{\circ}{\theta}_{1n} - k_\gamma \overset{\circ}{p}_{1n})\, \eta^{-1} S_n(\eta) / S_n(1) \tag{3.5}$$

If $n = 0$ the determinant of the system vanishes. The solutions may be determined, up to a solution of the homogeneous system, from Eq.(2.6) and the consistency condition (3.2):

$$w_{10}{}^\circ = k_s{}^2 k_\gamma^{-1} k_\lambda a_0 (a_0 + \delta_\sigma)^{-1} \varphi_0{}^\circ, \quad \theta_{10}{}^\circ = k_s^{-1} k_\gamma k_\lambda^{-1} w_{10}{}^\circ$$
$$j_{10}{}^\circ = -w_{10}{}^\circ, \quad p_{10}{}^\circ = \varphi_0{}^\circ, \quad a_{10}{}^\circ = a_{10}{}^\circ (t_1, t_2, \ldots) \tag{3.6}$$

By (3.2) and (3.3), if $\varphi_0{}^\circ \neq 0$ then, depending on the sign of $\varphi_0{}^\circ$, the bubble will either grow without limit or shrink to zero on the average. This is actually due to the fact that the pressure oscillates about the non-equilibrium pressure $p_*(1 + \varepsilon \varphi_0{}^\circ)$. But if $\varphi_0{}^\circ = 0$ (the system is on the average at equilibrium), the bubble radius in the first approximation will oscillate steadily $(\partial a_0 / \partial t_1 = 0)$, the average radius of the bubble varying on a "slower" time scale than t_1 (non-linear effect).

4. *Dynamics of the average bubble radius.* Let $\varphi_0{}^\circ = 0$. Then to a first approximation all unknown functions except perhaps $a_{10}{}^\circ$ are independent of t_1. We shall assume that $a_{10}{}^\circ = 0$, letting a_0 have the meaning of the average radius. Hence, using (1.1), (1.2) and (2.1)-(2.3), we obtain the inhomogeneities $f_2{}^{(\alpha)}$ in the second approximation ($m = 2$):

$$f_2^{(a)} = -\varkappa_g^{-1} a_0 \partial a_0 / \partial t_2, \quad f_2^{(j)} = a_1 j_1 \tag{4.1}$$

$$f_2^{(w)} = -\tau_w\left[a_1\partial w_1/\partial t_0 + \left(2\partial a_1/\partial t_0 - \frac{1}{2}\varkappa_g a_0^{-2} w_1\right)w_1\right]$$

$$f_2^{(\theta)} = a_0 d_\sigma^{-1}\left\{\left(\frac{1}{2}k_s^{-1} + \beta_T - \frac{3}{2}\right)\theta_1^2 + \left[\beta_p - k_s\left(\beta_T - \frac{1}{2}\right)\right]p_1\theta_1 - k_s\beta_p p_1^2\right\}$$

$$f_2^{(p)} = -\frac{a_0}{\varkappa_g}\frac{\partial a_0}{\partial t_2} - \frac{a_0^2}{\varkappa_g}\left[(a_1 + p_1)\frac{\partial a_1}{\partial t_0} + \frac{2a_1}{3\gamma}\frac{\partial p_1}{\partial t_0}\right] + (a_1 + \theta_1)j_1$$

$$f_2^{(l)} = \frac{a_0^2}{\varkappa_g}\left\{\left[\left(\eta - \frac{1}{\eta^2}\right)\frac{\partial a_1}{\partial t_0} + \frac{k_\rho\varkappa_g}{a_0^2\eta^2}j_1\right]\frac{\partial u_1^{(l)}}{\partial\eta} - 2a_1\frac{\partial u_1^{(l)}}{\partial t_0}\right\}$$

$$f_2^{(g)} = \frac{u_1^{(g)}}{\eta^2}\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial u_1^{(g)}}{\partial\eta}\right) - \left(\frac{\partial u_1^{(g)}}{\partial\eta}\right)^2 + \frac{a_0^2}{\varkappa_g}\left\{\left(\frac{\partial a_1}{\partial t_0} + \frac{1}{3\gamma}\frac{\partial p_1}{\partial t_0}\right)\eta\frac{\partial u_1}{\partial\eta} - \left[(p_1 + 2a_1)\frac{\partial u_1^{(g)}}{\partial t_0} - k_\gamma(u_1^{(g)} + 2a_1)\frac{\partial p_1}{\partial t_0}\right]\right\}$$

$$\beta_p = \frac{p_*}{\beta_*}\frac{\partial\beta}{\partial p}\bigg|_{p_*,\,T_*}, \quad \beta_T = \frac{T_*}{\beta_*}\frac{\partial\beta}{\partial T}\bigg|_{p_*,\,T_*}$$

where β_p, β_T are the non-dimensional derivatives of the accommodation coefficient with respect to the pressure and temperature, respectively, which characterize the "non-linear inhomogeneity" of the phase transition.

A sufficient condition for system (2.3) to have t_0-periodic solutions when $m = 2$ is that the zero harmonics $f_{20}^{(\alpha)}$ satisfy condition (2.7). The zero harmonic of the product of two t_0-periodic functions $x^{(1)}$ and $x^{(2)}$ with means $x_0^{(1)\circ} = x_0^{(2)\circ} = 0$ is computed from the definition (2.4):

$$(x^{(1)}x^{(2)})_0^\circ = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} x^{(1)}x^{(2)}\,dt_0 = \frac{1}{2}\sum_{n=1}^{\infty}\operatorname{Re}\{\overline{x_n^{(1)\circ}}\,x_n^{(2)\circ}\}$$

where the bar denotes complex conjugation.

Evaluating the appropriate integrals with respect to η, we can write condition (2.7) as a differential equation:

$$(a_0 + \delta_\sigma)\frac{\partial a_0}{\partial t_2} = \frac{\varkappa_g k_s}{2k_\gamma k_\lambda}\sum_{n=1}^{\infty}\left\{-\frac{\varkappa_g\tau_w k_s}{2a_0^2}|s_n\mathring{a}_{1n}|^2 + \left(\frac{1}{2k_s} + \beta_T - \frac{3}{2}\right)|\mathring{\theta}_{1n}|^2 - k_s\beta_p|\mathring{p}_{1n}|^2 + \operatorname{Re}\left\{\frac{k_\gamma}{k_s k_\lambda}\left(1 - \frac{k_s}{k_\gamma} + \frac{2\sigma_\sigma}{a_0}\right)\mathring{a}_{1n}\overline{\mathring{j}_{1n}} - \frac{1}{k_\lambda}\left(1 - \frac{2}{3\gamma}\right)s_n\mathring{a}_{1n}\overline{\mathring{p}_{1n}} + (\beta_p - k_s\beta_T)\mathring{p}_{1n}\overline{\mathring{\theta}_{1n}} + \left[\frac{1}{k_\lambda} - \frac{k_\rho}{k_\varkappa}(1 - E_n)\right]\mathring{\theta}_{1n}\overline{\mathring{j}_{1n}} + \frac{s_n}{k_n}E_n\mathring{\theta}_{1n}\overline{\mathring{a}_{1n}}\right\}\right\} \tag{4.2}$$

$$E_n = e^{\sqrt{s_n/k_\varkappa}}E_3(\sqrt{s_n/k_\varkappa}), \quad E_3(z) = \int_1^\infty e^{-z\zeta}\zeta^{-3}\,d\zeta$$

If we assume that the pressure field is monochromatic $(\varphi_1^\circ = 1,\ \varphi_n^\circ = 0,\ n > 1)$, the phase transitions are quasi-equilibrium $(d_\sigma \to 0)$ and the bubble is temperature-homogeneous $(h_n^{(g)} = 1/_3 s_n$, corresponding to the principal term $h_n^{(g)}$ of the asymptotic expansion as $s_n \to 0)$, Eqs.(4.2), (3.4) agree with the equation for the dynamics of the average bubble radius obtained in /4, 5/.

Figs.1and 2 present the results of computations with Eq.(4.2) for a system with the thermophysical parameters of water and vapour at a pressure $p_* = 0.1$ MPa we have used the notation $v = (a_0/\varkappa_g)\,\partial a_0/\partial t_2$, $A = a_0(\omega/\varkappa_g)^{1/2}$. Lacking reliable figures for the accommodation coefficient, the curves shown in the figures were computed at the following β_* values: 4×10^{-4} (the dotted curves), 0.04 (the solid curves), and the 0.4 (dash-dot curves); the dashed curves correspond to a quasi-equilibrium phase transition scheme. The "non-linear inhomogeneity" parameters in these cases were taken as $\beta_p = 0$, $\beta_T = 1.5$, in accordance with the theoretical formula of Landau /4/. It was found that variation of β_T within reasonable limits had little effect on the computed results. The frequency $\nu = \omega/(2\pi)$ of the acoustic field, which was assumed to be monochromatic, was 10 kHz.

A phase portrait of Eq.(4.2) is shown in Fig.1. The range of variation of a_0 was chosen to allow for the restrictions on the applicability of our model. The resonance values at which the growth rate of the bubble increases markedly are clearly visible. In the range of greater values the growth rate of the average bubble radius falls sharply. One interesting observation is that at certain radii the growth rate of the average radius is not a monotonic function of the accommodation coefficient; this effect is particularly evident in the neighbourhood of the resonance radii. The explanation is that reducing the accommodation coefficient not only reduces the phase transition intensity ($\beta = 0$ indicates no phase transitions), but also increases the amplitudes of the oscillations and phase shifts among the fluctuations of pressure, temperature and radius; the computations show that these factors taken in concert make a positive contribution to the growth rate of the average radius (see (3.4), (4.2). Yet another peculiarity of the phase portrait is the existence of a stable zero in the region of large A (steady radius) which, in the range of A values examined, was observed only at $\beta_* = 4 \times 10^{-4}$ ($a_{st} \sim 1$ cm; see Fig.1). The existence of a steady radius in the region of super-resonant radii was noted in /4/.

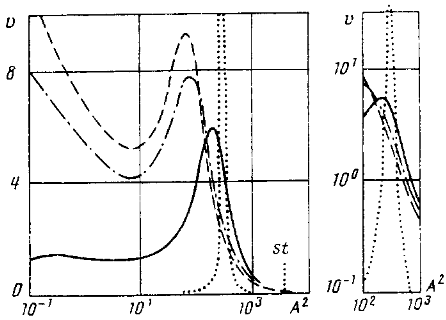

Fig.1

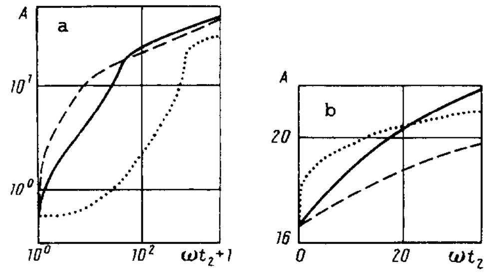

Fig.2

Fig.2 illustrates the numerical solution of the Cauchy problem for Eq.(4.2). The curves for $\beta_* = 0.4$ are pratically the same as that corresponding to the quasi-equilibrium phase transition and are not shown in the figure. The relative positions of curves with different β values depend on the initial condition $a_{00} = a_0(0)$, as is clear from the phase portrait in Fig.1. In case a $a_{00} = 10$ μm and in case b $a_{00} = 0.3$ mm.

If the thickness of the unsteady temperature boundary layer in the gas is much less than the bubble radius ($\delta_g = [\varkappa_g/(2\omega)]^{1/2} \ll a_0$), there will generally be a thin boundary layer in the liquid, since $\varkappa_l < \varkappa_g$. Using asymptotic representations of the functions in (3.4) and (4.2) for large $|s_n|^{1/2}$, we obtain the following equation (the high-frequency approximation):

$$(a_0 + \delta_\sigma)\frac{\partial a_0}{\partial t_2} = \frac{k_s \varkappa_g}{2k_\varkappa} \sum_{n=1}^{\infty} \frac{|\varphi_n^\circ|^2}{|D_n|^2} \left\{ -\frac{k_s k_\lambda}{6k_\gamma(\gamma-1)} \frac{a_0^2}{a_r^2(\omega n)} + \sqrt{\frac{|s_n|}{2}} H(\sqrt{\omega_\sigma n}) \right\} \tag{4.3}$$

$$|D_n|^2 = |1 - a_0^2 a_r^{-2}(\omega n) + i\omega n \tau_w k_\gamma \sqrt{|s_n|/2}(1+i) I(\sqrt{\omega_\sigma n})|$$

$$H(x) = P(x)/Q(x), \quad I(x) = R(x)/S(x), \quad \omega_\sigma = d_\sigma^2/\delta_g^2$$

$$R(x) = 1 + [(1+i)/\sqrt{2}] Kx, \quad S(x) = c_0 + [(1+i)/\sqrt{2}] c_1 x$$

$$P(x) = {}^1/_2 b_2 x^2 + b_1 x + b_0, \quad Q(x) = {}^1/_2 K^2 x^2 + Kx + 1$$

$$c_0 = 1 + k_\gamma^s (K-2), \quad c_1 = K - k_\gamma^s, \quad b_2 = c_1 K, \quad k_\gamma^s = k_3/k_\gamma$$

$$b_1 = k_\gamma^s K^2 + [(1 - k_\gamma^s)^2 - (k_\gamma^s)^2(2 - k_c)] K + (k_\gamma^s)^2 (1 - k_c)$$

$$b_2 = k_\gamma^s [1 - k_\gamma^s (1 - k_c)] K + (1 - k_\gamma^s)^2 - (k_\gamma^s)^2 k_c, \quad k_c = c_l/c_g$$

$$K = k_\gamma^s (1 + k_\lambda k_\varkappa^{-1/2}), \quad a_r(\omega) = \omega^{-1}(3\gamma p_*/\rho_l)^{1/2}$$

Here the first (negative) term in the term in curly brackets, whose contribution becomes dominant for at large radii or high frequencies, represents a drop of the average pressure in the bubble below its equilibrium value, due to the non-linearity of the Rayleigh-Lamb equation (i.e., the liquid at infinity is, as it were, underheated). The second (positive) term describes the dissipation of energy due to the phase transition in the problem with a plane boundary (the function H is analogous to that introduced in /3/ for high-frequency oscillations of a drop). The overall result of these effects is the appearance of a stable steady radius, which depends on the frequency and shape of the oscillations. As follows from (4.3), in the case of a monochromatic field with $\delta_g \ll a_0$ the steady average radius is

$$a_{st}(\omega) = 9(\gamma-1)^2 [k_s k_\lambda \omega \tau_w]^{-1} [\omega/(2\varkappa_g)]^{-1/2} H(\omega_\sigma^{1/2})$$

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DETERMINATION OF THE DRAG ON OSCILLATING PLATES IN A FLUID*

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Using an approximate approach /1/, methods of determining the vortex drag on plates undergoing harmonic oscillations in an incompressible fluid are considered. By means of this approach, the problem can be reduced to determining the velocity intensity coefficients (VIC's) on the edges of the plates and computing a certain integral over the boundary contour. Mathematically, the VIC's are analogous to the stress intensity coefficients (SIC's) /2/ in destruction mechanics. The most important exact solutions and closed expressions for the VIC's are presented for the planar and the spatial problems. To obtain numerical solutions, a version of the direct boundary-element method (BEM) is developed. Examples of applications of the finite-element method (FEM) and the BEM to specific problems are given. Methods for improving the accuracy of the numerical solutions are proposed. The results of experimental investigations are presented and compared with the computations.

1. Formulation of the problem. Consider the oscillations of a plate in an incompressible fluid at rest at long distances. We introduce the following notation: R is the characteristic linear dimension of the plate, v_0 and ω are the characteristic velocity amplitude and oscillation frequency of the plate, ρ and ν are the density and the kinematic viscosity of the fluid, and $\mathrm{Re} = v_0R/\nu$ and $\mathrm{Sh} = R\omega/v_0$ are the Reynolds and Strouhal numbers. We shall assume that the condition

$$\mathrm{Re}^{-1/2} \ll \mathrm{Sh}^{-3/4} \ll 1 \tag{1.1}$$

is satisfied.

The condition establishes a relation between the orders of magnitude of the thickness of the oscillating boundary layer, the dimensions of the eddy domain in the vicinity of the sharp edges, and the dimensions of the plate /1/. Outside small domains of essential eddies the motion of the fluid will be assumed to be a potential one.

We represent the velocity potential of the fluid in the form $\Phi(\mathbf{r}, t) = \varphi(\mathbf{r}) \cos \omega t$, where $\mathbf{r}$ is the position vector of a point and t is the time.

We have the boundary condition $\partial\varphi/\partial n = \mp v_n(\mathbf{r})$ on the surface of the plate. The "minus" and "plus" signs correspond to the "positive" and "negative" sides of the plate, which is assumed to be infinitely thin and $\mathbf{n}$ denotes the outer normal unit vector.

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